

Multiply Phased Traveling BPS Vortex

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We present the multiply phased current carrying vortex solutions in the $U(1)$ gauge theory coupled to an $(N+1)$ -component $SU(N+1)$ scalar multiplet in the Bogomolny limit. Our vortex solutions correspond to the static vortex dressed with traveling waves along the axis of symmetry. What is notable in our vortex solutions is that the frequencies of traveling waves in each component of the scalar field can have different values. The energy of the static vortex is proportional to the topological charge of CP^N model in the BPS limit, and the multiple phase of the vortex supplies additional energy contribution which is proportional to the Noether charge associated to the remaining symmetry.

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I. INTRODUCTION

Topological string-like vortex solutions have been the subject of intense studies in field theory and cosmology since the discovery of the Nielsen-Olesen (NO) string in the Abelian Higgs model [1]. Their existence is closely connected to the spontaneous breaking of the $U(1)$ gauge symmetry by Higgs mechanism. The string-like solutions have been shown to exist when the theory is generalized to have a global $SU(2)$ symmetry with a scalar doublet [2, 3]. This is interesting because this model represents the bosonic sector of the electroweak theory in the limit that the Weinberg angle, $\sin^2 \theta_W$, becomes one.

One of the most interesting properties of this solution is that its stability is guaranteed without being supported by the standard topological argument related to the homotopy of the vacuum manifold. In the Bogomolny-Prasad-Sommerfield (BPS) limit where the mass ratio of gauge field and the Higgs become equal, there is a continuous family of solutions which are degenerate in energy [4]. The solution space consists of either the embedded NO vortices or the baby skyrmion solutions.

In recent years, new type of solutions were found in this model. These twisted “semilocal” vortex were constructed allowing the two scalar components to have different phase factors which depend on time as well as on the coordinate corresponding to their axis of symmetry [5]. A notable fact about the twisted vortex is that the energy away from the BPS limit can be lower than the energy of the corresponding embedded NO vortex.

Clearly this type of vortex solutions is very interesting from the mathematical point of view. Moreover, from

the physical point of view they also become interesting because they have potentially important applications in various areas of physics. Obviously they could play important roles in condensed matter physics, in particular in multi-gap superconductors and multi-component Bose-Einstein condensates [6, 7]. This is because they are a natural generalization of the Abrikosov vortex in Ginzburg-Landau model of superconductor.

Moreover, they may have a natural application to the Skyrme theory, because the Skyrme theory also has the global $SU(2)$ and local $U(1)$ symmetry [8]. And they could play important roles in high energy physics because they could be embedded in the standard Weinberg-Salam model. Finally, they could describe cosmic strings and become important in cosmology. So they have interesting applications in almost all areas of physics.

The aim of this paper is to show that the multiply phased vortex solutions exist when we generalize the global $SU(2)$ to $SU(N+1)$. Especially we find the multiply phased vortex solutions in the BPS limit. The solutions can be obtained by dressing the traveling wave which moves along the axis of symmetry. What is remarkable in our vortex solutions is that the frequencies of traveling waves in each component of the $SU(N+1)$ scalar multiplet can be different, so that they are multiply phased.

The energy of the static BPS vortex is proportional to the topological charge of the CP^N model, and the multiple phase (or the “twisting”) of vortex supplies additional energy contribution which is proportional to the Noether charge associated to the remaining symmetry. We have analyzed this additional energy contributions in terms of known static BPS solutions and electromagnetic charge density in details.

The paper is organized as follows. In Section II we investigate the static BPS semilocal vortex solutions when the scalar field has $N+1$ components. We express all relevant equations in a gauge invariant way. In Section III

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we construct the multiply phased vortex solutions which have traveling waves along their axis of symmetry. We show the energy of multiply phased BPS vortex can be written in terms of topological charge of CP^N model and Noether charge. In the final section we discuss the physical implications of our results. In this paper we use the metric where $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

II. STATIC BPS VORTEX

The so-called “semi-local” string appears naturally in models involving multiply charged complex scalar fields coupled to electromagnetism which has a spontaneous symmetry breaking. They were introduced as a minimal extension of the Abelian Higgs model with the familiar topologically stable Nielsen-Olesen vortex.

In this extended model, the Abelian gauge field couples to a complex Higgs doublet so that in addition to local $U(1)$ symmetry, there is an extra $SU(2)$ global symmetry, which is spontaneously broken to a global $U(1)$. According to the popular wisdom the trivial first homotopy of vacuum manifold (i.e., $\pi_1(S^3) = 0$) rejects the existence of topological vortex, but when the scalar field at the spatial infinity lies on a gauge orbit or a circle lying in the vacuum manifold, the gradient energy of scalar field could have a finite value and thus the theory could support finite energy solitons. This leads to $U(1)$ vortex solutions even though the vacuum manifold is simply connected. They have important applications in cosmology, multi-component superconductor in condensed matter physics [9], and in two-component plasma physics [10].

Recently, it was shown that the semilocal vortex can carry persistent longitudinal currents associated with the global symmetry subgroup [5]. Our primary interest in the model is these properties of BPS semilocal vortex solutions. We found that it is possible to have a vortex with traveling wave whose frequencies are all different when the scalar field has more than two components.

Consider the extended Abelian Higgs model in which an $SU(N+1)$ multiplet scalar field $\phi = (\phi_1, \phi_2, \dots, \phi_{N+1})$ is coupled to the Abelian gauge field A_μ ,

$$\mathcal{L} = \frac{1}{2}|D_\mu\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - v^2)^2 - \frac{1}{4}(F_{\mu\nu})^2, \quad (1)$$

where $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Abelian field strength.

The Lagrangian possesses the local $U(1)$ gauge symmetry, being invariant under the transformations, $\delta\phi = i\alpha\phi$, $\delta A_\mu = -(1/e)\partial_\mu\alpha$, where $\alpha(x)$ is an infinitesimal parameter. Moreover, it has the global $SU(N+1)$ symmetry $\delta\phi = i\alpha_a T_a\phi$, where T^a ($a = 1, \dots, N+1$) are the generators of $SU(N+1)$ in the fundamental representation, and α_a are constant infinitesimal parameters.

When the scalar get a vacuum expectation values v the gauge symmetry is spontaneously broken. In the symmetry broken phase, the particle spectrum consists of a massive vector field with mass $m_A = ev$, and of scalar

Higgs with mass $m_H = \sqrt{\lambda}v$. The remainings of the particle spectrum are $2N$ -massless Goldstone bosons. For convenience we will rescale the fields so that all variables become dimensionless,

$$\begin{aligned} \phi &\rightarrow v\phi, & A_\mu &\rightarrow vA_\mu, \\ x_\mu &\rightarrow x_\mu/(ev), \end{aligned} \quad (2)$$

in the following.

Introducing CP^N field ξ ,

$$\phi = \rho\xi, \quad (\xi^\dagger\xi = 1), \quad (3)$$

the Lagrangian (1) can be written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2|D_\mu\xi|^2 - \frac{\beta}{8}(\rho^2 - 1)^2 \\ & - \frac{1}{4}(F_{\mu\nu})^2, \end{aligned} \quad (4)$$

where now the covariant derivative becomes $D_\mu\xi = (\partial_\mu + iA_\mu)\xi$ and $\beta = \lambda/e^2$. In terms of ρ and ξ the conserved current associated with the local $U(1)$ gauge symmetry can be written as

$$\begin{aligned} J_\mu &= \frac{1}{2i}\rho^2(\xi^\dagger D_\mu\xi - (D_\mu\xi)^\dagger\xi) \\ &= \rho^2(a_\mu + A_\mu), \end{aligned} \quad (5)$$

where $a_\mu = -i\xi^\dagger\partial_\mu\xi$ is the auxiliary $U(1)$ gauge field of CP^N model. With this J_μ which is gauge invariant we can express our equations in a gauge independent way.

Since

$$\begin{aligned} |D_\mu\xi|^2 &= |\nabla_\mu\xi|^2 + \frac{J_\mu^2}{\rho^4}, \\ \nabla_\mu\xi &= (\partial_\mu - ia_\mu)\xi = (\partial_\mu - \xi^\dagger\partial_\mu\xi)\xi, \end{aligned} \quad (6)$$

we can rewrite the Lagrangian in terms of gauge invariant quantities [11],

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2|\nabla_\mu\xi|^2 + \frac{J_\mu^2}{2\rho^2} - \frac{\beta}{8}(\rho^2 - 1)^2 \\ & - \frac{1}{4}(F_{\mu\nu})^2. \end{aligned} \quad (7)$$

The equations of motion are then given as

$$\partial^\mu F_{\mu\nu} = -J_\nu, \quad (8)$$

$$\partial_\mu^2\rho - |\nabla_\mu\xi|^2\rho - \frac{J_\mu^2}{\rho^3} + \frac{\beta}{2}(\rho^2 - 1)\rho = 0, \quad (9)$$

$$\nabla_\mu^2\xi + 2\left(\frac{\partial_\mu\rho}{\rho} + i\frac{J_\mu}{\rho^2}\right)\nabla_\mu\xi = (\xi^\dagger\nabla_\mu^2\xi)\xi. \quad (10)$$

The energy momentum tensor can be obtained varying the Lagrangian with respect to the metric tensor $g_{\mu\nu}$,

$$\begin{aligned} T_{\mu\nu} = & (\partial_\mu\rho)(\partial_\nu\rho) + \frac{1}{2}(\nabla_\mu\xi)^\dagger\nabla_\nu\xi + \frac{1}{2}(\nabla_\nu\xi)^\dagger\nabla_\mu\xi \\ & + \frac{J_\mu J_\nu}{\rho^2} - F_{\mu\kappa}F_{\nu\kappa} - g_{\mu\nu}\mathcal{L}. \end{aligned} \quad (11)$$

It is well-known that this model with scalar doublet can have finite energy (per unit length) vortex solutions called the semilocal vortices. Some of the solutions in this model are just those obtained from the Nielsen-Olesen solutions identifying, say, the upper component of ϕ with the scalar field in the Nielsen-Olesen model. But there are different kinds of solutions. Especially, at the critical coupling there exist a solution which appears to describe a hybrid of a NO vortex and CP^1 lump in addition to the embedded NO solution [3].

Now, we review the static straight vortex solutions whose axis is perpendicular to (x_1, x_2) plane. The energy (per unit length) of static vortex configuration reads

$$E = \int d^2x \left[\frac{1}{2}(\partial_i \rho)^2 + \frac{1}{2}\rho^2 |\nabla_i \xi|^2 + \frac{\beta}{8}(\rho^2 - 1)^2 + \frac{J_i^2}{2\rho^2} + \frac{1}{2}B^2 \right], \quad (12)$$

with $i = 1, 2$ and $B = F_{12} = \epsilon_{ij} \partial_i A_j$ is the component of magnetic field perpendicular to (x_1, x_2) plane. If one now chooses the coupling constants to satisfy

$$\beta = \frac{\lambda}{e^2} = 1, \quad (13)$$

the we can find a bound for the energy. At this critical value the two mass scales in the theory, scalar mass $m_H = \sqrt{\lambda}v$ and gauge field mass $m_A = ev$, are equal. This defines the Bogomolny limit in the Abelian Higgs model [4].

In order to show the Bogomolny limit in the current model, we will start by rearranging the term in the energy functional as follows

$$\begin{aligned} \frac{1}{2}(\partial_i \rho)^2 + \frac{J_i^2}{2\rho^2} &= \frac{1}{2} \left(\frac{J_i}{\rho} \pm \epsilon_{ik} \partial_k \rho \right)^2 \mp \epsilon_{ik} J_i \partial_k \ln \rho, \\ \frac{1}{2} |\nabla_i \xi|^2 &= \frac{1}{4} |\nabla_i \xi \pm i \epsilon_{ik} \nabla_k \xi|^2 \pm \frac{1}{2} f_{12}, \\ \frac{B^2}{2} + \frac{1}{8}(\rho^2 - 1)^2 &= \frac{1}{2} \left[B \mp \frac{1}{2}(\rho^2 - 1) \right]^2 \mp \frac{1}{2} B \\ &\quad \pm \frac{1}{2} \partial_i (\epsilon_{ik} J_k) \mp \epsilon_{ik} J_i \partial_k \ln \rho \mp \frac{\rho^2}{2} f_{12}, \end{aligned}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength of the auxiliary CP^N gauge field a_μ .

After making use of this relations the energy of vortex can be written as

$$E = \int d^2x \left\{ \frac{1}{2} \left(\frac{J_i}{\rho} \pm \epsilon_{ik} \partial_k \rho \right)^2 + \frac{1}{4} \rho^2 |\nabla_i \xi \pm i \epsilon_{ik} \nabla_k \xi|^2 + \frac{1}{2} \left(B \mp \frac{1}{2}(\rho^2 - 1) \right)^2 \mp \frac{1}{2} B \right\}. \quad (14)$$

From the last expression, we secure the Bogomolny bound

$$E \geq \frac{1}{2} |\Phi|, \quad (\Phi = \int d^2x B), \quad (15)$$

since the we can choose upper sign or lower sign in (14), depending on the sign of the flux Φ . For a given value of Φ , this bound is saturated if and only if the fields satisfy the self-duality equations

$$\nabla_i \xi \pm i \epsilon_{ik} \nabla_k \xi = 0, \quad (16)$$

$$J_i \pm \frac{1}{2} \epsilon_{ik} \partial_k \rho^2 = 0, \quad (17)$$

$$B \mp \frac{1}{2}(\rho^2 - 1) = 0. \quad (18)$$

Any solution to the first order self-duality equations automatically satisfy the original static field equations. In what follows we shall focus on vortex solutions (the upper sign) without loss of the generality.

Let us now discuss about the nature of these self-dual soliton solutions. The last two equations (17) and (18) can be combined to give a single second order differential equation

$$\partial_i^2 \ln \rho - \frac{1}{2}(\rho^2 - 1) - f_{12} = 0. \quad (19)$$

Note that the topological charge of CP^N model is defined by

$$T = \frac{1}{2\pi i} \int d^2x \epsilon_{ij} (\nabla_i \xi)^\dagger \nabla_j \xi = \frac{1}{2\pi} \int d^2x f_{12}. \quad (20)$$

For an embedded NO vortex solution the scalar field ρ should vanish at the location of vortex in order to have well defined phase. In this case we have

$$f_{12} = 2\pi \sum_{r=1}^n \delta(\vec{x} - \vec{x}_r),$$

where $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ are the locations of vortex core on the plane.

Thus the magnetic flux of NO vortex measures the vorticity

$$\int d^2x B = 2\pi n.$$

When $f_{12} = 0$, the solution of Eq. (19) describes a embedded self-dual NO vortex configuration. In this case the scalar field ρ has to vanish at the origin in order to have a well defined phase of vortex solution. Near the vortex cores we have

$$\partial_i^2 \log \rho = 2\pi n \delta^2(\vec{r}),$$

for n superimposed vortex. Furthermore, with (II) and (17) we can show that

$$\begin{aligned} \int d^2x eB &= \int d^2x \partial_i^2 \log \rho - \int d^2x f_{12} \\ &= 2\pi r \frac{d \ln \rho}{dr} \Big|_0^\infty - 2\pi T, \end{aligned} \quad (21)$$

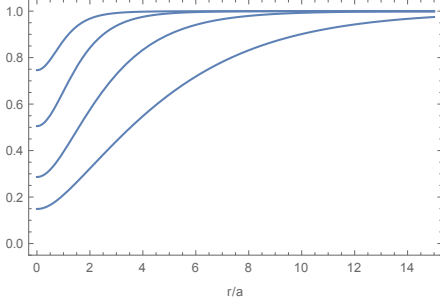


FIG. 1: Profiles of ρ for $k = 1$ with $a = 0.25, 0.5, 1, 2$. Notice that $\rho(0)$ is monotonically increasing as a becomes larger.

Since $\rho(r) = \rho_0 r^n + \dots$ for the embedded NO vortex, the magnetic flux has quantized value $|e \int d^2 x B| = 2\pi n$. When the scalar field ρ has no zeros, then the magnetic flux is solely expressed in terms of the topological charge of CP^N model,

$$\int d^2 x B = 2\pi T. \quad (22)$$

There are two kind of degenerate solutions depending on the value of scalar field ρ at the origin, embedded NO vortex and dubbed baby skyrmions [2, 3]. From now on we will concentrate on the self-dual baby skyrmion.

The most general solution for Eq. (16) can be given in terms of N (anti-)holomorphic functions u_a ($a = 1, 2, \dots, N$),

$$\xi = \frac{1}{\sqrt{1 + |u|^2}} \begin{pmatrix} u \\ 1 \end{pmatrix}. \quad (23)$$

where $u = (u_1, u_2, \dots, u_N)^T$ is N column vector. The topological charge density can be written as

$$\begin{aligned} f_{12} &= \frac{1}{2} \partial_i^2 \ln(1 + |u|^2) = \frac{2(u')^\dagger \Delta^2 u'}{(1 + |u|^2)^2} \\ &= \frac{2|\Delta u'|^2}{(1 + |u|^2)^2}, \end{aligned} \quad (24)$$

where u' means the derivative of u with respect to its argument and Δ^2 is a $N \times N$ matrix defined by

$$\Delta^2 = (1 + |u|^2)I_N - u \otimes u^\dagger, \quad (25)$$

where I_N is a $N \times N$ identity matrix.

Let us consider the simplest CP^1 rotationally symmetric lump solution, $u = u_1 = (a/z)^k$ where $z = x_1 + ix_2$ and a is the width parameter of soliton. Since CP^1 model is scale invariant, the width parameter a cannot be fixed. The topological charge of this lump solution is given as $T = k$, and $\Delta^2 = 1$. With a radial coordinate $s = r/a$ rescaled with respect to the width parameter a of CP^1 lump, the condensate equation (19) becomes

$$\frac{1}{s} \frac{d}{ds} \left(s \frac{d}{ds} \ln \rho \right) - \frac{a^2}{2} (\rho^2 - 1) = \frac{2k^2 s^{2k-2}}{(1 + s^{2k})^2} \quad (26)$$

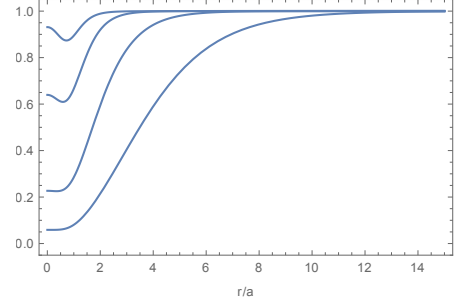


FIG. 2: Profiles of ρ for $k = 2$, with $a = 0.5, 1, 2, 4$. Here again ρ increases monotonically.

Although solution in the BPS limit is characterized with parameter a the energy is independent of the width of solution. This implies the existence of a zero model associated with the width of semilocal vortex.

For small values of s ,

$$\rho(s) = b \left[1 + \left(\frac{\delta_{k1}}{2} - \frac{a^2}{8} + \frac{a^2 b^2}{8} \right) s^2 + \dots \right]. \quad (27)$$

For a given CP^1 lump size a , we can find the solution ρ which goes to the vacuum as $s \rightarrow \infty$ with appropriate choice of $\rho(0) = b$ as a shooting parameter. The asymptotic behavior of condensate ρ is

$$\rho(s) = \left(1 - \frac{2}{s^{2k}} + \dots + C \frac{e^{-as}}{\sqrt{s}} + \dots \right), \quad (28)$$

Fig. 1 and Fig. 2 show the profiles of ρ for $k = 1, 2$ for various width parameter values.

Since the magnetic field is related with the ρ by Eq. (18), we can see that magnetic field will show power-law decay asymptotically. This feature is in sharp contrast with the embedded NO vortex where the magnetic field is confined within the size determined by the mass of gauge field.

III. MULTIPLY PHASED BPS VORTEX

Now, let us investigate semilocal vortex solutions with space-time dependent phases. Requiring vortex solution to have translational symmetry in time and along the vortex axis, x_3 , the scalar fields can get a phase which is linearly dependent on t and x_3 . In particular, we are interested in vortex solutions with traveling wave type phases along its axis x_3 ,

$$\begin{aligned} \phi &= \rho(r) \xi, \\ \xi &= \frac{1}{\sqrt{1 + |u|^2}} \begin{pmatrix} u_1(z) e^{i\omega_1(t-x_3)} \\ u_2(z) e^{i\omega_2(t-x_3)} \\ \vdots \\ u_N(z) e^{i\omega_N(t-x_3)} \\ 1 \end{pmatrix}, \\ A_\mu &= A_\mu(x_1, x_2), \end{aligned} \quad (29)$$

where the gauge field is assumed to have no t, x_3 -dependency. Note that the frequencies of traveling wave in each component of scalar field are all distinct values. Thus there are relative phases (twist of phase) between any two components of scalar field. This feature cannot be realized in the semilocal model with scalar doublet. A similar construction of CP^N lump solutions dressed with traveling waves in the ungauged CP^N model has been considered in [13].

This t and x_3 dependency in twisted vortex ansatz gives nontrivial a_0 and a_3 components of the auxiliary field:

$$a_0 = -a_3 = \frac{u^\dagger \Omega u}{1 + |u|^2}, \quad (30)$$

where

$$\Omega = \text{diag}(\omega_1, \omega_2, \dots, \omega_N). \quad (31)$$

Again this permits a twisted vortex to have nonzero charge density J_0 and a longitudinal current density J_3 , which is obvious from (II).

With the relations

$$\begin{aligned} (\nabla_0 + \nabla_3)\xi &= 0, \\ (\nabla_0^2 - \nabla_3^2)\xi &= 0, \end{aligned} \quad (32)$$

one can show that the twisted vortex ansatz becomes a solution of equations of motion (8), (9), and (10) when the following condition is met,

$$J_0 = -J_3. \quad (33)$$

With this relation the self-duality equations (16), (17), and (18) defined in (x_1, x_2) plane are completely decoupled from the those in (t, x_3) directions. If the wave were traveling down along the x_3 -axis, then we should have $J_0 = +J_3$. Both the cases satisfy the relation $J_0^2 = J_3^2$. This implies that there is a longitudinal electric current J_3 carried by traveling waves and the region with nonvanishing current is electrically charged.

From (8) we can see that the charge density J_0 is governed by the following equation,

$$\partial_i^2 \left(\frac{J_0}{\rho^2} \right) - \partial_i f_{i0} = J_0. \quad (34)$$

The second term can be expressed in terms of u explicitly:

$$\begin{aligned} \partial_i f_{i0} &= \frac{4(u')^\dagger \Delta^2 \Omega u'}{(1 + |u|^2)^2} - \frac{4(u')^\dagger \Delta \Omega u}{(1 + |u|^2)^3} (u^\dagger u') \\ &\quad - \frac{4(u')^\dagger \Delta^2 u}{(1 + |u|^2)^3} (u^\dagger \Omega u). \end{aligned} \quad (35)$$

Thus with the solution of (19) we can solve Eq. (34) in principle.

If all traveling waves have the same frequency, it will give a much simpler vortex solution. For generality, we

will discuss solutions in which each traveling wave in the a -th CP^N component has a distinct frequency.

Note, the left hand side of (34) can be written

$$\partial_i^2 \left(\frac{J_0}{\rho^2} - a_0 \right) = J_0. \quad (36)$$

Integrating both sides of this over the transverse plane gives vanishing net charge per unit vortex length,

$$q_{\text{tot}} = \int d^2 x J_0 = 0 \quad (37)$$

This also implies the vanishment of net longitudinal current because of the relation $J_0^2 = J_3^2$.

The ansatz (29) breaks the original global $SU(N+1)$ symmetry to N $U(1)$. Under this residual global symmetry the fields ξ transform as

$$\begin{aligned} \delta \xi_a &= i\alpha_a \xi_a, \quad a = 1, 2, \dots, N, \quad (\text{no sum over } a), \\ \delta \xi_{N+1} &= 0. \end{aligned} \quad (38)$$

The Noether currents associated with these symmetries are given by

$$\mathcal{J}_\mu^{(a)} = \rho^2 (\xi_a^\dagger (D_\mu \xi)_a - (D_\mu \xi)_a^\dagger \xi_a), \quad (39)$$

where no summation is implied over $a = 1, 2, \dots, N$. Since $\partial_0 \xi_a = i\omega_a \xi_a$ and $\partial_0 \xi_{N+1} = 0$, we get the following relation

$$\sum_{a=1}^N \omega_a \mathcal{Q}^{(a)} = \int d^2 x \left(\rho^2 |\nabla_0 \xi|^2 + a_0 J_0 \right), \quad (40)$$

where $\mathcal{Q}^{(a)}$ is the Noether charge per unit length of the vortex

$$\mathcal{Q}^{(a)} = \int d^2 x \mathcal{J}_0^{(a)}. \quad (41)$$

In the following we will show that the twisted vortex carry these global Noether charges.

The tension of twisted vortex (energy per unit length) is given by

$$\begin{aligned} E &= \pi T + E_{\text{twist}}, \\ E_{\text{twist}} &= \int d^2 x \left(\frac{1}{2} \rho^2 |\nabla_0 \xi|^2 + \frac{1}{2} \rho^2 |\nabla_3 \xi|^2 \right. \\ &\quad \left. + \frac{J_0^2}{2\rho^2} + \frac{J_3^2}{2\rho^2} + \frac{1}{2} E_i^2 + \frac{1}{2} B_i^2 \right), \end{aligned} \quad (42)$$

where $E_i = F_{i0}$ and $B_i = \epsilon_{ij} F_{j3}$, ($i, j = x_1, x_2$) are the transverse components of electric and magnetic field respectively. From (II) we have

$$E_i = \partial_i \left(\frac{J_0}{\rho^2} - a_0 \right) = -\epsilon_{ij} E_j, \quad (43)$$

and with (34) the energy of transverse components of electric field can be expressed as

$$\int d^2 x E_i^2 = \int d^2 x \left(-\frac{J_0^2}{\rho^2} + a_0 J_0 \right). \quad (44)$$

Since $\partial_0 \xi = -\partial_3 \xi$ and $J_0 = -J_3$ the transverse components of magnetic field are given as

$$B_i = -\epsilon_{ij} E_j. \quad (45)$$

Thus the energy contribution from twisting of vortex can be expressed with the Noether charges,

$$\begin{aligned} E_{\text{twist}} &= \int d^2x \left(\rho^2 |\nabla_0 \xi|^2 + \frac{J_0^2}{\rho^2} + E_i^2 \right) \\ &= \sum_{a=1}^N \omega_a Q^{(a)}. \end{aligned} \quad (46)$$

For a embedded ANO vortex or skyrmion solutions correspond to untwisted ($\omega_k = 0$) static solutions.

This shows that the energy of a twisted BPS vortex of the type (29) depends on the global Noether charges in addition to the topological charge,

$$E = \pi T + \sum_{a=1}^N \omega_a Q^{(a)}. \quad (47)$$

This is unlike the case of untwisted semilocal vortex where the Bogomolny bound is given in terms of the topological charge only.

Let us now discuss the topological properties of the E_{twist} . E_{twist} can be into three parts as have done in Ref. [14]. Each parts can be expressed with scalar field ρ determined from (19), CP^N lump solution $w(z)$, and charge density J_0 determined from (34).

With (43) and (34) we have the following relation

$$\int d^2x \left(E_i^2 + \frac{J_0^2}{\rho^2} \right) = \int d^2x \left(\frac{J_0}{\rho^2} \partial_i f_{i0} + f_{i0}^2 \right). \quad (48)$$

We can obtain the similar relations for J_3 , f_{i3} , and B_i . Using these we have the following expression for the energy contribution from twisting. Thus the energy contribution from twisting consists of three separate contributions.

$$E_{\text{twist}} = \int d^2x \left(\rho^2 |\nabla_0 \xi|^2 + f_{i0}^2 + \frac{J_0}{\rho^2} \partial_i f_{i0} \right). \quad (49)$$

Firstly, note that

$$\epsilon_1 = \int d^2x \rho^2 |\nabla_0 \xi|^2 = \int d^2x \frac{|\Delta \Omega u|^2}{(1+|u|^2)^2} \rho^2. \quad (50)$$

where

$$\Omega = \text{diag}(\omega_1, \omega_2, \dots, \omega_N). \quad (51)$$

If the frequencies of all traveling waves are equal, $\Omega = \omega_0 I_N$, then we have $|\Delta \Omega u|^2 = \omega_0^2 |u|^2$ so that the integral is proportional to ω_0^2 . However, for a configuration with unit topological charge this term diverges logarithmically since as $r \rightarrow \infty$, $\rho \rightarrow \text{const}$, and $|u| \rightarrow 1/r$ as have noted in Ref. [15].

The second contribution is given by

$$\epsilon_2 = \int d^2x f_{i0}^2 = \int d^2x \frac{|u^\dagger \Omega \Delta^2 u|^2}{(1+|u|^2)^4}, \quad (52)$$

which can be evaluated if a explicit form of u were given. This integral is also proportional to ω_0^2 if all frequencies of waves are equal to ω_0 .

Remaining contribution to energy can be written as

$$\begin{aligned} \epsilon_3 &= \int d^2x \frac{J_0}{\rho^2} \partial_i f_{i0} \\ &= \frac{4}{i} \int d^2x \left\{ \frac{(u')^\dagger \Delta^2 \Omega u'}{(1+|u|^2)^2} - \frac{((u')^\dagger \Delta^2 \Omega u)(u^\dagger u')}{(1+|u|^2)^3} \right. \\ &\quad \left. - \frac{|\Delta u'|^2 (u^\dagger \Omega u)}{(1+|u|^2)^3} \right\} \frac{J_0}{\rho^2}, \end{aligned} \quad (53)$$

where we used the relation (II). Again if all traveling waves has the same frequency ω_0 , then one can notice from Eq. (34) that $\partial_i f_{i0}$ is proportional to ω_0 . This implies that J_0/ρ^2 is proportional to ω_0 and the last integral is proportional to ω_0^2 . Thus twisting the BPS semilocal vortex gives additional energy contributions which depend on the frequencies Ω as well as topological charge T :

$$E = \pi T + \epsilon_1(T, \Omega) + \epsilon_2(T, \Omega) + \epsilon_3(T, \Omega). \quad (54)$$

Since the ansatz (29) moves along the gauge orbit of the unbroken global $U(1)^N$ as one moves along the x_3 axis, $\partial_3 \xi_a = -i\omega_a \xi_a$, ($a = 1, 2, \dots, N$), we expect the translational symmetry along x_3 axis. Consequently a twisted vortex should carry the conserved longitudinal momentum

$$\begin{aligned} P &= \int d^2x T_0^3 \\ &= \int d^2x \left(F_{0i} F_{3i} - \frac{\rho^2}{2} (\nabla_0 \xi)^\dagger (\nabla_3 \xi) \right. \\ &\quad \left. - \frac{\rho^2}{2} (\nabla_3 \xi)^\dagger (\nabla_0 \xi) - \frac{J_0 J_3}{\rho} \right). \end{aligned} \quad (55)$$

Indeed, it can be written in terms of the Noether charges as the corresponding conserved longitudinal momentum (per unit length) $P = T_0^3$ carried by a twisted vortex also can be expressed in terms of Noether charges,

$$\begin{aligned} P &= \int d^2x \left(F_{0i} F_{3i} + \frac{\rho^2}{2} (\nabla_0 \xi)^\dagger (\nabla_3 \xi) \right. \\ &\quad \left. + \frac{\rho^2}{2} (\nabla_3 \xi)^\dagger (\nabla_0 \xi) + \frac{J_0 J_3}{\rho^2} \right) \sum_{a=1}^N \omega_a Q^{(a)}. \end{aligned} \quad (56)$$

Imposing the rotational symmetry to the twisting vortex solutions the conserved angular momentum of the vortex will be also conserved.

Since the gauge field of our ansatz has no x_3 axis and the x_3 dependent phases in scalar field are compensated by a gauge transformation, the solution has a translation invariance along x_3 axis. The associated conserved

momentum (per unit length) is given by

$$P = \int d^2x T_z^0. \quad (57)$$

Our twisting solution has the nontrivial longitudinal current density flowing through any transverse plane of the vortex. Integrating the Eq. (34) over the (x_1, x_2) plane shows that total current, and total charge density per unit vortex length are both zero,

$$I_3 = \int d^2x J_3 = - \int d^2x J_0 = -I_0 = 0. \quad (58)$$

IV. SUMMARY

In this paper we have presented twisted BPS vortex solutions in the extended Abelian Higgs model where the scalar field is in the fundamental representation of $SU(N+1)$ group. This twisted vortex solution has different phases in each components of scalar multiplet. The phases are linear in x_3 and t . Twisting static vortex

introduces an additional energy contributions which are related with the conserved Noether charge of the remaining symmetry. They have the conserved longitudinal momentum which can be expressed by the Noether charge.

It will be interesting to check whether it is possible to support the traveling wave of the form $e^{i(at \mp bx_3)}$ with $a^2 - b^2 \neq 0$ in the current model.

From the mathematical point of view our vortex solutions are interesting. Moreover, from the physical point of view they have potentially important applications in various areas of physics, in high energy physics, cosmology, and condensed matter physics. In particular they could have important roles in multi-gap superconductors and multi-component Bose-Einstein condensates. This is because they are a natural generalization of the Abrikosov vortex in Ginzburg-Landau model of superconductor.

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